Available online at www.sciencedirect.com
SCIENCE DIRECT
PHYSICS LETTERS B
Physics Letters B 610 (2005) 141–146
www.elsevier.com/locate/physletb

Trouble with space-like noncommutative field theory

V. Gayral [a,1], J.M. Gracia-Bondía [b], F. Ruiz Ruiz [b]

[a] *Centre de Physique Théorique, UMR 6207, 13288 Marseille, France*
[b] *Departamento de Física Teórica I, Universidad Complutense de Madrid, 28040 Madrid, Spain*

Received 7 January 2005; accepted 3 February 2005

Editor: L. Alvarez-Gaumé

Dedicated to A. Galindo on his 70th birthday

Abstract

It is argued that the one-loop effective action for space-like noncommutative (i) $\lambda\varphi^4$ scalar field theory and (ii) $U(1)$ gauge theory does not exist. This indicates that such theories are not renormalizable already at one-loop order and suggests supersymmetrization and reinvestigating other types of noncommutativity.

PACS: 11.15.-q; 11.30.Pb; 11.10.Gh

Keywords: Noncommutative field theory; Renormalization; Non-local UV divergences

The consistency of a noncommutative quantum field model crucially depends, even at one loop, on the choice of the matrix $\Theta := (\Theta^{\mu\nu})$ of noncommutative parameters. As is well known, in a noncommutative field theory with superficial UV divergences, one-loop radiative corrections to 1PI Green functions consist of a planar part and a nonplanar part. The planar part contains the local UV divergences and has the same dependence on Θ as the classical action, whereas the nonplanar part is UV-finite, but depends on Θ in a complicated way. By adding suitable local counterterms to the classical action, the UV divergences in the planar parts are subtracted and one is thus left with renormalized Green functions.

These functions must be consistent with unitarity. However, combining renormalizability and unitarity is not a trivial undertaking. It is known that for space-like or 'magnetic' Θ the one-loop renormalized Green functions preserve unitarity [1]. Indeed, since the time–space components Θ^{0i} of a space-like Θ vanish, no violations of unitarity are introduced when using Wick rotation to compute Feynman integrals. The problem that arises then is that

E-mail addresses: gayral@cpt.univ-mrs.fr (V. Gayral), ferruiz@fis.ucm.es (F. Ruiz Ruiz).
[1] Also at Université de Provence.

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2005.02.011

the renormalized 1PI Green functions, while being consistent with unitarity, in general do not yield a well-defined effective action, thus putting in jeopardy the existence of space-like noncommutative quantum field theories. In this Letter we show that this is the case for a real scalar field theory and $U(1)$ gauge theory, and discuss the connections with other issues in the literature.

We derive this result in configuration space, in which it comes out in a very elegant and natural way, and then translate it into terms of the known expressions for the 1PI Green functions in momentum space. It is important to mention at the outset that we are able to recover for the latter functions the results already in the literature. The point is that they do not define an effective action.

Let us consider a real field $\varphi(x)$ with classical action

$$S[\varphi] = \int d^4x \left(-\frac{1}{2}\varphi\Box\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi\star\varphi\star\varphi\star\varphi \right),$$

where $\Box = \partial^\mu\partial_\mu$ is the d'Alambertian, m^2 is the mass squared, λ is the coupling constant and $\star$ denotes the Moyal product. We consider space-like noncommutativity, for which the components Θ^{0i} vanish $(i = 1, 2, 3)$. Without loss of generality, by a rotation, Θ can be put in the canonical form

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \theta S \end{pmatrix}, \quad \text{with } S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is then convenient to split the coordinates x of a point in spacetime as $x = (\tilde{x}, \bar{x})$, with $\tilde{x} = (x^0, x^1)$ and $\bar{x} = (x^2, x^3)$. Similarly, we write $p = (\tilde{p}, \bar{p})$ in momentum space. Here the Moyal product of two functions f and g is given by

$$f \star g(x) = \frac{1}{(2\pi)^2} \int d^2\bar{u}\, d^2\bar{v}\, e^{-i\bar{u}\cdot\bar{v}} f\left(\tilde{x}, \bar{x} - \tfrac{1}{2}\theta S\bar{u}\right) g(\tilde{x}, \bar{x} + \bar{v}). \tag{1}$$

This definition, a particular case of one due to Rieffel [2], is equivalent to the more familiar one

$$f \star g(x) = f(x) \exp\left(\frac{i}{2}\theta\varepsilon^{\mu\nu} \overleftarrow{\partial_{\bar{x}^\mu}} \overrightarrow{\partial_{\bar{x}^\nu}} \right) g(x),$$

under conditions spelled in [3].

Using standard techniques and after Wick rotating the time coordinate, the one-loop contribution Γ_1 to the effective action can be recast [4] as

$$\Gamma_1[\varphi] = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\, \mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right),$$

where the operators H and H_0 are given by

$$H = H_0 + V \quad \text{with } H_0 = \Delta + m^2 \text{ and } V = \frac{\lambda}{6}(L_{\varphi\star\varphi} + R_{\varphi\star\varphi} + L_\varphi R_\varphi).$$

Here $\Delta = -\partial^\mu\partial_\mu$ is the Laplacian in four-dimensional Euclidean space and L_f and R_f denote the operators of left and right Moyal multiplication by f. Note that, as we have defined it, the Laplacian Δ is positive-definite, also that L_f and R_f commute with each other, and that V becomes in the commutative limit the ordinary multiplication operator by $\lambda\varphi^2/2$. The expression for Γ_1 above needs regularization. We regularize it by using a zeta regularization-like method which amounts to replacing $1/t$ in the integrand with $\mu^{2\epsilon}/t^{1-\epsilon}$, where ϵ is complex and μ is a mass scale introduced to keep the correct mass dimension. Thus we write for the regularized one-loop contribution to the effective action

$$\Gamma_1^{\mathrm{reg}}[\varphi] = -\frac{\mu^{2\epsilon}}{2}\int_0^\infty \frac{dt}{t^{1-\epsilon}}\, \mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right).$$

Our concern here is to calculate the divergent part of $\Gamma_1^{\mathrm{reg}}[\varphi]$ as $\epsilon \to 0$, to see whether the resulting divergences can be subtracted by local counterterms, and if the result of doing so is well-defined. To calculate the divergent parts as $\epsilon \to 0$, we proceed in two steps. First, we use the covariant perturbation method of Barvinsky and Vilkovisky [5] to compute

$$\Gamma_1^{\mathrm{reg}}[\varphi] = -\frac{\mu^{2\epsilon}}{2} \int_0^\infty \frac{dt}{t^{1-\epsilon}} \sum_{n=1}^{\infty} K_n(\Delta, V, t), \tag{2}$$

where $K_n(\Delta, V, t)$ is given by

$$K_n(\Delta, V, t) = \frac{(-t)^n}{n} \int_0^1 d\alpha_1 \cdots d\alpha_n \, \delta\left(1 - \sum_1^n \alpha_i\right) \mathrm{Tr}\big[V \mathrm{e}^{-t\alpha_1 H_0} V \mathrm{e}^{-t\alpha_2 H_0} \cdots V \mathrm{e}^{-t\alpha_n H_0}\big].$$

Secondly, we retain in the expansion (2) the terms potentially divergent at $\epsilon = 0$. Using that $K_n(\Delta, V, t) \sim \mathrm{e}^{-tm^2} t^{n-2}$ as $t \to 0$ (for precise estimates we refer to [6, Section 10.2]), one immediately sees that for $n \geqslant 3$ the corresponding contribution to the effective action is finite, and that potential divergences only occur for $n = 1$ and $n = 2$. This is as expected, for $n = 1$ and $n = 2$ correspond to 2- and 4-point 1PI Green functions, known to be superficially divergent, whereas terms with $n \geqslant 3$ correspond to 1PI Green functions with six or more φ, and these are known to be finite. We thus write

$$\Gamma_1^{\mathrm{reg}}[\varphi] = \frac{\mu^{2\epsilon}}{2} \int_0^\infty \frac{dt}{t^{1-\epsilon}} \mathrm{Tr}\left[tV \mathrm{e}^{-tH_0} - \frac{t^2}{2} \int_0^1 d\alpha \, V \mathrm{e}^{-t\alpha H_0} V \mathrm{e}^{-t(1-\alpha)H_0} + O\big(\varphi^6\big)\right]. \tag{3}$$

Let us concentrate on the contribution to the regularized effective action quadratic in φ, given by the first term in (3). Using the expressions of H_0 and V, it takes the form

$$\Gamma_1^{\mathrm{reg}(2)}[\varphi] = \frac{\lambda \mu^{2\epsilon}}{12} \int_0^\infty dt \, t^\epsilon \mathrm{e}^{-tm^2} \, \mathrm{Tr}\big[(L_{\varphi\star\varphi} + R_{\varphi\star\varphi} + L_\varphi R_\varphi)\mathrm{e}^{-t\Delta}\big]. \tag{4}$$

Writing the trace as

$$\mathrm{Tr}(\cdots) = \int d^4x \, \langle x| \cdots |x\rangle,$$

using Rieffel's expression (1) for the star product and recalling

$$\langle x|\mathrm{e}^{-t\Delta}|y\rangle = \frac{1}{(4\pi t)^2} \mathrm{e}^{-|x-y|^2/4t},$$

it is straightforward to show that the first and second terms in the parenthesis in (4) give the same contribution

$$\mathrm{Tr}\big(L_{\varphi\star\varphi} \mathrm{e}^{-t\Delta}\big) = \mathrm{Tr}\big(R_{\varphi\star\varphi} \mathrm{e}^{-t\Delta}\big) = \frac{1}{(4\pi t)^2} \int d^4x \, \varphi^2(x), \tag{5}$$

and that the third one yields

$$\mathrm{Tr}\big(L_\varphi R_\varphi \mathrm{e}^{-t\Delta}\big) = \int d^4x \int \frac{d^2\bar{u}}{(2\pi\theta)^2} \varphi(\tilde{x}, \bar{x}) \varphi(\tilde{x}, \bar{x} + \bar{u}) \frac{\mathrm{e}^{-t\bar{u}^2/\theta^2}}{4\pi t}. \tag{6}$$

It is clear that the contributions to the effective action $\Gamma_1^{\mathrm{reg}(2)}[\varphi]$ that result from these traces are well-defined if ϵ is not an integer equal or less than one. In fact, keeping ϵ away from the poles and integrating over t, we obtain

$$\Gamma_1^{\mathrm{reg}(2)}[\varphi] = \frac{\lambda m^2}{96\pi^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\Gamma(-1+\epsilon)\int d^4x\,\varphi^2(x) + \frac{\lambda}{192\pi^3\theta^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\Gamma(\epsilon)\int d^4x\int d^2\bar{u}\,\varphi(\tilde{x},\bar{x})\varphi(\tilde{x},\bar{x}+\bar{u})\left(1+\frac{\bar{u}^2}{m^2\theta^2}\right)^{-\epsilon}. \tag{7}$$

In the first term we recognize a planar *local* contribution. Hence, the divergence that occurs in it as $\epsilon \to 0$ can be subtracted by adding a local counterterm to the classical action. This is the usual mass renormalization. The second contribution, however, is nonplanar and develops a *nonlocal singularity* as $\epsilon \to 0$ that cannot be subtracted by a local counterterm. This means that the theory is not renormalizable.

Let us understand this result in terms of 1PI Green functions. By functionally differentiating the regularized effective action we generate the regularized 1PI 2-point Green function $\Gamma_1^{\mathrm{reg}(2)}(x_1, x_2)$ as the sum

$$\Gamma_1^{\mathrm{reg}(2)}(x_1, x_2) = \left.\frac{\delta^2\Gamma_1^{\mathrm{reg}(2)}[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi=0} = \Gamma_{1,\mathrm{P}}^{\mathrm{reg}(2)}(x_1-x_2) + \Gamma_{1,\mathrm{NP}}^{\mathrm{reg}(2)}(x_1-x_2)$$

of a planar part

$$\Gamma_{1,\mathrm{P}}^{\mathrm{reg}(2)}(x_1-x_2) = \frac{\lambda m^2}{48\pi^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\Gamma(-1+\epsilon)\delta^{(4)}(x_1-x_2)$$

and a nonplanar part

$$\Gamma_{1,\mathrm{NP}}^{\mathrm{reg}(2)}(x_1-x_2) = \frac{\lambda}{96\pi^3\theta^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\Gamma(\epsilon)\delta^{(2)}(\tilde{x}_1-\tilde{x}_2)\left[1+\frac{(\bar{x}_1-\bar{x}_2)^2}{m^2\theta^2}\right]^{-\epsilon}.$$

The nonplanar part is neither finite nor local at $\epsilon = 0$. This represents a particularly nasty strain of UV divergences in the nonplanar sector and makes the theory unrenormalizable already at one loop.

We next show how this result fits with the expressions in the literature for one-loop radiative corrections in momentum space. To this end, we need the Fourier transform of the 2-point 1PI Green function obtained above. Restricting ourselves to the problematic nonplanar part, one finds when ϵ is not a negative integer [7] for its Fourier transform

$$\Sigma_{1,\mathrm{NP}}^{\mathrm{reg}}(\tilde{p},\bar{p}) := \int d^4z\,\mathrm{e}^{-ipz}\,\Gamma_{1,\mathrm{NP}}^{\mathrm{reg}(2)}(z) = \frac{\lambda m^2}{24\pi^2}\left(\frac{\mu^2}{2m^2}\right)^{\epsilon}\frac{K_{1-\epsilon}(\theta m|\bar{p}|)}{(\theta m|\bar{p}|)^{1-\epsilon}}.$$

Here $K_\nu(\cdot)$ denotes the third Bessel function of order ν. Apparently the limit $\epsilon \to 0$ of this expression is harmless and one is tempted to write

$$\Sigma_{1,\mathrm{NP}}(\tilde{p},\bar{p}) = \frac{\lambda m^2}{24\pi^2}\frac{K_1(\theta m|\bar{p}|)}{\theta m|\bar{p}|},$$

which after rotating back to Minkowski momentum space, reproduces indeed the known results in the literature—see, e.g., [8]. The key point, however, is that the function $K_1(\theta m|\bar{p}|)/\theta m|\bar{p}|$ is not locally integrable and thus it does not define a distribution; it has no Fourier transform either. This matters because in the integral

$$\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\Sigma_{1,\mathrm{NP}}(\tilde{p},\bar{p})\hat{\varphi}(p)\hat{\varphi}(-p), \tag{8}$$

which would define the corresponding contribution to the effective action, $\hat{\varphi}(p)$ is a c-number function [9, Section 6.2] and such integral is undefined.

Let us discuss this point in some more detail. For a generic c-number function $\hat{\varphi}(p)$ the integral (8) does not exist, since

$$\frac{K_1(\theta m|\bar{p}|)}{\theta m|\bar{p}|}d^2\bar{p} \sim \frac{d|\bar{p}|}{|\bar{p}|} \quad \text{for } |\bar{p}| \to 0.$$

This hints that a way to avoid the problem would be for the 2-point 1PI Green function to depend on a linear combination of $|\tilde{p}|$ and $|\bar{p}|$ so that $\Sigma_{1,\mathrm{NP}}$ would only diverge for both $|\tilde{p}|$ and $|\bar{p}| \to 0$, in which case the modulus $|p|$ of the four-momentum approaches zero and the measure $d^4p \sim |p|^3 d|p|$ improves the convergence of the integral (8). To have such a dependence, the rank of the noncommutativity matrix must be four instead of two. The reason for this is that covariance implies that the 2-point 1PI Green function in momentum space may only depend on the mass and $p \circ p := p_\mu \Theta^\mu_\nu \Theta^{\nu\sigma} p_\sigma$, and for $p \circ p$ to involve $|\tilde{p}|$ and $|\bar{p}|$, a rank-four Θ matrix is needed. Indeed, if the noncommutativity matrix has the form

$$\Theta = \begin{pmatrix} \zeta S & 0 \\ 0 & \theta S \end{pmatrix},$$

$p \circ p$ takes the form $\zeta^2|\tilde{p}|^2 + \theta^2|\bar{p}|^2$. The 2-point 1PI Green function in configuration euclidean space can then be calculated along the very same lines as presented here, with the same result for the planar part, whereas for the nonplanar part one obtains

$$\Gamma^{\mathrm{reg}(2)}_{1,\mathrm{NP}}(x_1 - x_2) = \frac{\lambda(\mu^2\theta^2\zeta^2)^\epsilon}{96\pi^4} \frac{\Gamma(1+\epsilon)}{(\theta^2\zeta^2 m^2 + \theta^2|\tilde{x}_1 - \tilde{x}_2|^2 + \zeta^2|\bar{x}_1 - \bar{x}_2|^2)^{1+\epsilon}}.$$

See Ref. [10] for the same calculation with a different approximation method for the heat kernel and a different regularization scheme. Now the effective action and the Fourier transform

$$\Sigma^{\mathrm{reg}}_{1,\mathrm{NP}}(\tilde{p}, \bar{p}) = \frac{\lambda m^2}{24\pi^2}\left(\frac{\mu^2}{2m^2}\right)^\epsilon \frac{K_{1-\epsilon}(m\sqrt{p \circ p})}{(m\sqrt{p \circ p})^{1-\epsilon}}$$

exist for $\epsilon = 0$. The latter depends on $|\tilde{p}|$ and $|\bar{p}|$, thus making the integral (8) well-defined. Furthermore, this formula can be Wick-rotated to Minkowski momentum space.

Our result holds as well for noncommutative $U(1)$ gauge theory. To see this, recall [11] on the one hand that for noncommutative $U(1)$ gauge theory in an arbitrary Lorentz gauge the nonplanar part of the 2-point 1PI Green function in Minkowski momentum space behaves for arbitrary $\Theta \to 0$ as

$$i\Pi^{\mu\nu}_{1,\mathrm{NP}}(\tilde{p}, \bar{p}) = \frac{2i}{\pi^2}\frac{(\Theta p)_\mu(\Theta p)_\nu}{(p \circ p)^2} + \frac{i}{16\pi^2}\left(\frac{13}{3} - \alpha\right)\ln\bigl(p^2 p \circ p\bigr)\bigl(p^2 g_{\mu\nu} - p_\mu p_\nu\bigr) + O\bigl(\Theta^0\bigr), \tag{9}$$

where α is the gauge-fixing parameter. On the other hand, on the grounds of covariance, we observe that $\Pi_{\mu\nu}$ depends on $\bar{p}$ only through $(\Theta p)_\mu$, p^2 and $p \circ p$. Hence, for a space-like Θ, for which $p \circ p = (\theta|\bar{p}|)^2$, the behaviour of $\Pi_{\mu\nu}(p)$ as $|\bar{p}| \to 0$ is given by its behaviour as $\theta \to 0$. It then follows from Eq. (9) that

$$\Pi^{\mu\nu}_{1,\mathrm{NP}}(\tilde{p}, \bar{p}) d^2\bar{p} \sim \frac{d|\bar{p}|}{|\bar{p}|} \quad \text{for } |\bar{p}| \to 0 \text{ and } \mu, \nu = 2, 3.$$

This proves once again that the integral

$$\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} A_\mu(p) \Pi^{\mu\nu}_{1,\mathrm{NP}}(\tilde{p}, \bar{p}) A_\nu(-p),$$

that would define the contribution to the effective action quadratic in the gauge field does not exist for $A^\mu(p)$ a c-number. The examples exhibited in this Letter suggest that this disease is quite general for space-like noncommutative field theories with UV/IR mixing. This result may help to explain the inability to define quasiplanar Wick products within the framework of the Yang–Feldman approach to quantum field theory [12].

In view of the state of affairs expounded here, several alternatives to quantum field theory with space-like noncommutativity are worth considering.

- Other types of noncommutativity should be further investigated in regard of unitarity. Some progress along this line has been made in Refs. [13] and [14].

- So far we have been discussing the 2-point part of the effective action. The 4-point part, given by the second term in (3), can also be evaluated in the same way, and turns out to be free of this disease (although not of the UV/IR mixing). On the face of it, one could think of performing by fiat a nonlocal renormalization of the 2-point function.
- The pathology uncovered in this paper adds itself to an endemic list of troubles [11,15]. If one wants to stay within the realm of conventional perturbative renormalization, it seems again that the only way out for space-like noncommutativity is to supersymmetrize the theory. In the case of the scalar theory, one would expect that, in the same manner as for gauge theories the quadratic noncommutative IR divergences in the 2-point 1PI Green functions are cancelled by the supersymmetric partners of the gauge field, the supersymmetric partners of the scalar field φ would cancel the nonintegrability in $\Sigma^{\mathrm{ren}}_{1,\mathrm{NP}}$ so as to render a well-defined contribution to the effective action.

We note finally that it has been reported [16] that, for noncommutative gauge theory with space-like noncommutativity, the axial anomaly acquires a non-planar contribution. A natural question is whether the disease found here is related to this anomaly phenomenon.

Acknowledgements

V.G. wishes to acknowledge the hospitality of the Department of Theoretical Physics of Universidad Complutense de Madrid. J.M.G.B. thanks MEC, Spain, for support through a ‘Ramón y Cajal’ contract. He also acknowledges discussions with E. López. F.R.R. is grateful to MEC, Spain, for financial support through grant BFM2002-00950.

References

[1] J. Gomis, T. Mehen, Nucl. Phys. B 591 (2000) 265.
[2] M.A. Rieffel, Memoirs Am. Math. Soc. 506 (1993).
[3] R. Estrada, J.M. Gracia-Bondía, J.C. Várilly, J. Math. Phys. 30 (1989) 2789.
[4] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon Press, Oxford, 1997.
[5] A.O. Barvinsky, G.A. Vilkovisky, Nucl. Phys. B 333 (1990) 471.
[6] J.M. Gracia-Bondía, J.C. Várilly, H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser, Boston, 2001.
[7] L. Schwartz, Theórie des Distributions, Hermann, Paris, 1966.
[8] F. Ruiz Ruiz, Nucl. Phys. B 637 (2002) 143.
[9] C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw–Hill, New York, 1980, Section 6.2.
[10] V. Gayral, hep-th/0412233.
[11] F. Ruiz Ruiz, Phys. Lett. B 502 (2001) 274.
[12] D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Phys. Rev. D 71 (2005) 025022, hep-th/0408204.
[13] D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Phys. Lett. B 533 (2002) 178.
[14] K. Fujikawa, hep-th/0410146.
[15] A. Matusis, L. Susskind, N. Toumbas, JHEP 0012 (2000) 002l;
L. Alvarez-Gaumé, M.A. Vázquez-Mozo, Nucl. Phys. B 668 (2003) 293.
[16] F. Ardalan, N. Sadooghi, Int. J. Mod. Phys. A 17 (2002) 123;
A. Armoni, E. López, S. Theissen, JHEP 0206 (2002) 050.